ABELIAN VARIETIES OVER FINITE FIELDS AS BASIC ABELIAN VARIETIES

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ABSTRACT. In this note we show that any basic abelian variety with additional structures over an arbitrary algebraically closed field of characteristic $p > 0$ is isogenous to another one defined over a finite field. We also show that the category of abelian varieties over finite fields up to isogeny can be embedded into the category of basic abelian varieties with suitable endomorphism structures. Using this connection, we derive a new mass formula for a finite orbit of polarized abelian surfaces over a finite field.

1. INTRODUCTION

In this note we work on abelian varieties over fields of characteristic $p > 0$, particularly on basic abelian varieties with additional structures (endomorphisms, a polarization and a level structure). Conceptually, an abelian variety with fixed additional structures is *basic* if the corresponding point in a moduli space of PEL-type over $\overline{\mathbb{F}}_p$ lands in the minimal Newton stratum (Rapoport-Zink [6] and Rapoport [5]). The group-theoretic definition was introduced by Kottwitz [1]. This notion is geometric in the sense that an abelian variety with additional structures is basic if and only if its base change to any algebraically closed field extension is also basic. As isogenous abelian varieties land in the same Newton stratum, an abelian variety with additional structures that is isogenous to a basic one is also basic.

Let B be a finite-dimensional semi-simple $\mathbb{Q}$-algebra with a positive involution $*$ and O_B an order in B stable under $*$. A polarized O_B-abelian variety is a triple (A, λ, ι) where A is an abelian variety with polarization λ and $\iota : O_B \to \mathrm{End}(A)$ is a ring monomorphism which is compatible with λ. We recall the definition of basic polarized O_B-abelian varieties (A, λ, ι) in Section 2.

Basic abelian varieties with additional structures share many similar properties with supersingular abelian varieties without additional structures, and many techniques employed there can be carried over here as well. For example, similar to supersingular abelian varieties, one can formulate a geometric mass for a finite orbit of basic abelian varieties and relate this geometric mass to an arithmetic mass defined by group theory. The well-known Deuring-Eichler mass formula is obtained in this fashion. We refer to [15] for more discussions in this aspect. In this paper we prove the following result, which may be regarded as another analogue property enjoyed by supersingular abelian varieties.

Theorem 1.1. *Let $\underline{A} = (A, \lambda, \iota)$ be a basic polarized O_B-abelian variety over an algebraically closed field k of characteristic $p > 0$. Then there exists a polarized*

Date: February 24, 2016.

O_B-abelian variety $\underline{A}' = (A', \lambda', \iota')$ over a finite field κ and an O_B-linear isogeny $\varphi : A' \otimes_\kappa k \to A$ over k that preserves the polarizations.

The second part of this note studies the converse to Theorem 1.1. We show that any abelian variety over a finite field can be regarded as a basic abelian variety with suitable endomorphism structures. More precisely, if A is an abelian variety over the finite field $\mathbb{F}_q$ of $q = p^s$ elements and $F = \mathbb{Q}(\pi_A) \subset \mathrm{End}(A) \otimes \mathbb{Q}$ is the $\mathbb{Q}$-subalgebra generated by its Frobenius endomorphism π_A, then the abelian variety A together with the F-action is a basic F-abelian variety (Proposition 4.1). See Remark 3.2 for the notion of a B-abelian variety being basic. A priori, the original definition of basic abelian varieties with additional structures requires both structures of endomorphisms and polarizations. However, similar to supersingular abelian varieties, polarizations play no role in the characterization of supersingularity.

Let $\mathcal{A}_{\mathbb{F}_q}$ denote the category of abelian varieties over $\mathbb{F}_q$ up to isogeny, and $\mathcal{B}^{\mathrm{rig}}$ be the category of basic abelian varieties with rigidified endomorphisms over $\overline{\mathbb{F}}_p$ up to isogeny, defined in Section 4. We prove the following result.

Theorem 1.2. *There is a functor Φ that embeds the category $\mathcal{A}_{\mathbb{F}_q}$ as a full subcategory of $\mathcal{B}^{\mathrm{rig}}$.*

Theorem 1.2 connects (polarized) abelian varieties over a finite field $\mathbb{F}_q$ with basic (polarized) F-abelian varieties over $\overline{\mathbb{F}}_p$ equipped with a suitable commutative semi-simple $\mathbb{Q}$-algebra F. This connection is particularly useful when the $\mathbb{Q}$-algebra F is fixed. In this case one may consider a smaller class of (polarized) abelian varieties over $\mathbb{F}_q$ whose endomorphism rings contain the maximal order O_F. This smaller set of isomorphism classes of polarized abelian varieties over $\mathbb{F}_q$ is embeddable into the basic locus of a moduli space of polarized O_F-abelian varieties; see Lemma 5.1 and (5.2). Below is a example where we use this embedding to derive a mass formula for a class of polarized abelian surfaces over $\mathbb{F}_p$.

Choose a simple abelian variety A_0 over the prime finite field $\mathbb{F}_p$ whose Frobenius endomorphism π_0 satisfies that $\pi_0^2 = p$. Then A_0 is a superspecial abelian surface, i.e. the base change $A_0 \otimes \overline{\mathbb{F}}_p$ is isomorphic to the product of two supersingular elliptic curves. Let us consider the set Λ of isomorphism classes of principally polarized simple abelian surfaces (A, λ) over $\mathbb{F}_p$ such that A is isogenous to A_0. Put $F = \mathbb{Q}(\pi_0) = \mathbb{Q}(\sqrt{p})$ and O_F its ring of integers. Let $\Lambda^{\max} \subset \Lambda$ be the subset of classes $[(A, \lambda)]$ such that $O_F \subset \mathrm{End}(A)$. We can show that $\Lambda^{\max}$ is a nonempty set. As usual, the mass $\mathrm{Mass}(\Lambda^{\max})$ of $\Lambda^{\max}$ is defined by

$$\mathrm{Mass}(\Lambda^{\max}) := \sum_{(A,\lambda)\in\Lambda^{\max}} |\,\mathrm{Aut}(A, \lambda)|^{-1}. \tag{1.1}$$

Then we show that $\mathrm{Mass}(\Lambda^{\max})$ is equal to the mass of a finite Hecke orbit S in the superspecial locus of a Hilbert modular surface modulo p. Furthermore, using the geometric mass formula for the superspecial orbits established in [12], we obtain the mass formula

$$\mathrm{Mass}(\Lambda^{\max}) = \frac{\zeta_F(-1)}{4}, \tag{1.2}$$

where $\zeta_F(s)$ the Dedekind zeta function of F (see Section 5.2).

The paper is organized as follows. In Section 2 we recall the definition of basic abelian varieties with additional structures. The proof and some consequences of

Theorem 1.1 are given in Section 3. In Section 4 we show that any abelian variety over a finite field, together with the action of the center of its endomorphism algebra, is a basic abelian variety. This result is used to construct the functor Φ in Theorem 1.2. In the last section we consider the isogeny class of simple supersingular abelian surfaces mentioned as above and compute the associated mass (1.1).

Notation. If M is a $\mathbb{Z}$-module or a $\mathbb{Q}$-module and ℓ is a prime, we write $M_\ell := M \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ or $M_\ell = M \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, respectively. For any perfect field k of characteristic $p > 0$, denote by $W(k)$ the ring of Witt vectors over k, $B(k)$ the field of fractions of $W(k)$, σ the Frobenius map on $W(k)$ and $B(k)$ induced by $\sigma : k \to k,\ x \mapsto x^p$. If F is a finite product of number fields F_i, denote by O_F the maximal order in F. A prime $\mathbf{p}$ of F over p, denoted by $\mathbf{p}|p$, means a prime of F_i for some F_i or a prime ideal of O_F over p. For an abelian variety A over a field k, write $\mathrm{End}(A) = \mathrm{End}_k(A)$ for the endomorphism ring of A over k and $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ for the endomorphism algebra of A over k. If A is defined over a finite field $\mathbb{F}_q$, we denote by π_A the Frobenius endomorphism of A over $\mathbb{F}_q$.

2. Basic abelian varieties with additional structures

In this section we recall the concept of basic abelian varieties with additional structures introduced by Kottwitz [1]. Our reference is Rapoport-Zink [6, p.11, p. 281 and 6.25, p. 291].

2.1. **Setting.** Let B be a finite-dimensional semi-simple algebra over $\mathbb{Q}$ with a positive involution $*$, and O_B be an arbitrary order of B stable under $*$.

Recall that a non-degenerate $\mathbb{Q}$-valued skew-Hermitian B-space is a pair (V, ψ) where V is a left faithful finite B-module, and $\psi : V \times V \to \mathbb{Q}$ is a non-degenerate alternating pairing such that $\psi(bx, y) = \psi(x, b^* y)$ for all $b \in B$ and all $x, y \in V$.

A *polarized O_B-abelian variety* (resp. *polarized B-abelian variety*) is a triple $\underline{A} = (A, \lambda, \iota)$, where (A, λ) is a polarized abelian variety and $\iota : O_B \to \mathrm{End}(A)$ (resp. $\iota : B \to \mathrm{End}^0(A)$) is a ring monomorphism such that $\lambda\iota(b^*) = \iota(b)^t\lambda$ for all $b \in O_B$. Here $\iota(b)^t : A^t \to A^t$ denotes the dual morphism of $\iota(b)$.

Let $\underline{A}$ be a polarized O_B-abelian variety over k, where k is an arbitrary field. For any prime ℓ (not necessarily invertible in k), we write $\underline{A}(\ell)$ for the associated ℓ-divisible group with additional structures $(A[\ell^\infty], \lambda_\ell, \iota_\ell)$, where λ_ℓ is the induced quasi-polarization from $A[\ell^\infty]$ to $A^t[\ell^\infty] = A[\ell^\infty]^t$ (the Serre dual), and $\iota_\ell : (O_B)_\ell \to \mathrm{End}(A[\ell^\infty])$ the induced ring monomorphism. If $\ell \neq \mathrm{char}(k)$, let $T_\ell(A)$ denote the ℓ-adic Tate module of A, $V_\ell := T_\ell(A) \otimes \mathbb{Q}_\ell$, and let

$$\rho_\ell : \mathcal{G}_k \to \mathrm{GU}_{B_\ell}(V_\ell, e_\lambda) \tag{2.1}$$

be the associated Galois representation. Here $\mathcal{G}_k := \mathrm{Gal}(k_s/k)$ is the Galois group of k, k_s a separably closure of k, and

$$\mathrm{GU}_{B_\ell}(V_\ell, e_\lambda) := \{ g \in \mathrm{Aut}_{B_\ell}(V_\ell) \,|\, e_\lambda(gx, gy) = c\, e_\lambda(x, y) \text{ for some } c \in \mathbb{Q}_\ell^\times \}$$

is the group of B_ℓ-linear similitudes with respect to the Weil pairing

$$e_\lambda = e_{\lambda,\ell} : T_\ell(A) \times T_\ell(A) \to \mathbb{Z}_\ell(1),$$

where

$$\mathbb{Z}_\ell(1) := \varprojlim \mu_{\ell^m}(k_s)$$

is the Tate twist.

If k is a perfect field of characteristic p, let $M(\underline{A})$ denote the covariant Dieudonné module of $\underline{A}$ with the additional structures and put $N(\underline{A}) := M(\underline{A}) \otimes_{W(k)} B(k)$, the rational Dieudonné module (or the isocrystal) with the additional structures; see [10, Section 1].

In this note we consider only the objects $\underline{A} = (A, \lambda, \iota)$ for which there is a non-degenerate skew-Hermitian B-space (V, ψ) with $2 \dim A = \dim_{\mathbb{Q}} V$. Namely, we require that there exists a complex polarized O_B-abelian variety with the same dimension as A. For example, we exclude the case where A is a supersingular elliptic curve and B is the quaternion $\mathbb{Q}$-algebra ramified precisely at $\{p, \infty\}$.

2.2. **Basic abelian varieties.** Let k be any field of characteristic p and $\bar{k}$ be an algebraic closure of k. Put $W := W(\bar{k})$ and $L := B(\bar{k})$. Let (V_p, ψ_p) be a $\mathbb{Q}_p$-valued non-degenerate skew-Hermitian B_p-module. A polarized O_B-abelian variety $\underline{A}$ over $\bar{k}$ is said to be *related to* (V_p, ψ_p) if there is a $(B_p \otimes_{\mathbb{Q}_p} L)$-linear isomorphism $\alpha : N(\underline{A}) \simeq (V_p, \psi_p) \otimes_{\mathbb{Q}_p} L$ which preserves the pairings for a suitable identification $L(1) \simeq L$.

Let $G_p := \mathrm{GU}_{B_p}(V_p, \psi_p)$ be the algebraic group over $\mathbb{Q}_p$ of B_p-linear similitudes with respect to the pairing ψ_p. A choice of α gives rise to an element $b \in G_p(L)$ by transport of structure of the Frobenius map on $N(\underline{A})$, that is, $\alpha : N(\underline{A}) \simeq (V_p \otimes L, b(\mathrm{id} \otimes \sigma), \psi_p)$ becomes an isomorphism of isocrystals with additional structures. The σ-conjugacy class $[b]$ of b in $G_p(L)$ is independent of the choice of α. The decomposition of $V_p \otimes L$ into isotypic components (the components of a single slope) induces a $\mathbb{Q}$-graded structure, and thus defines a (slope) homomorphism $\nu_b : \mathbf{D} \to G_p$ over some unramified finite extension $\mathbb{Q}_{p^s}$ of $\mathbb{Q}_p$, where $\mathbf{D}$ is the pro-torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$. The set $\nu_{[b]} = \{\nu_b\}$ for $b \in [b]$ is the $G_p(L)$-conjugacy class of ν_b for a single $b \in [b]$, called the Newton vector associated to $N(\underline{A})$.

Definition 2.1. (1) A polarized O_B-abelian variety $\underline{A}$ over $\bar{k}$ is said to be *basic with respect to* (V_p, ψ_p) if

(a) $\underline{A}$ is related to (V_p, ψ_p), and
(b) the slope homomorphism $\nu_b : \mathbf{D} \to G_p$ for $b \in [b]$ is central.

(2) The object $\underline{A}$ over $\bar{k}$ is said to be *basic* if it is basic with respect to (V_p, ψ_p) for some non-degenerate skew-Hermitian B_p-space (V_p, ψ_p).

(3) A polarized O_B-abelian variety $\underline{A}$ over any field k is said to be *basic* if its base change $\underline{A} \otimes_k \bar{k}$ is basic.

Clearly a polarized O_B-abelian variety $\underline{A}$ is basic if (and only if) it is so considered as polarized B-abelian variety. Two polarized B-abelian varieties $\underline{A}_1$ and $\underline{A}_2$ are said to be *isogenous*, denote $\underline{A}_1 \sim \underline{A}_2$, if there is a B-linear isogeny $\varphi : A_1 \to A_2$ such that the pull-back $\varphi^* \lambda_2$ is a $\mathbb{Q}$-multiple of λ_1. Clearly the property for an object $\underline{A}$ being basic is an isogeny invariant property. From the definition it is also easy to see that this is a geometric notion: an object $\underline{A} = (A, \lambda, \iota)$ over k is basic if and only if the base change $\underline{A} \otimes_k k_1$ is basic for any algebraically closed field $k_1 \supset k$.

3. Proof of Theorems 1.1 and its corollaries

3.1. To prove Theorem 1.1, we need some properties of basic abelian varieties with additional structures. Let (V, ψ) be a non-degenerate ($\mathbb{Q}$-valued) skew-Hermitian

B-space and let $G := \mathrm{GU}_B(V, \psi)$ be the algebraic group over $\mathbb{Q}$ of B-linear similitudes with respect to the pairing ψ.

Let F be the center of B and F_0 be the $\mathbb{Q}$-subalgebra fixed by the induced involution on F, which we denote by $a \mapsto \bar{a}$. Let Σ_p be the set of primes $\mathbf{p}$ of F over p, and for each prime $\mathbf{p}|p$, denote by $\mathrm{ord}_{\mathbf{p}}$ the corresponding p-adic valuation normalized in a way that $\mathrm{ord}_{\mathbf{p}}(p) = 1$. Let $F_p := F \otimes \mathbb{Q}_p = \prod_{\mathbf{p}|p} F_{\mathbf{p}}$ be the decomposition into a product of local fields. For each isocrystal N with an F_p-linear action, let

$$N = \oplus_{\mathbf{p}|p} N_{\mathbf{p}} \tag{3.1}$$

be the decomposition with respect to the F_p-action.

Lemma 3.1 (Rapoport-Zink)**.** *Let the notation be as above.*

(1) *The center Z of G is the algebraic subgroup over $\mathbb{Q}$ whose group of R-points is*

$$Z(R) = \{x \in (F \otimes R)^\times;\, x\bar{x} \in R^\times\},$$

for any $\mathbb{Q}$-algebra R.

(2) *Let N be an isocrystal with additional structures and suppose that it is related to $(V \otimes \mathbb{Q}_p, \psi)$. Then N is basic with respect to $(V \otimes \mathbb{Q}_p, \psi)$ if and only if each component $N_{\mathbf{p}}$ is isotypic. In particular, if N is basic, then $N_{\mathbf{p}}$ is supersingular for primes $\mathbf{p}$ with $\mathbf{p} = \bar{\mathbf{p}}$.*

Proof. Statement (1) and the only if part of statement (2) are proved in 6.25 of [6]. The if part is easier: as each $N_{\mathbf{p}}$ is isotypic, say of slope $r_{\mathbf{p}}/s$, the slope homomorphism $s\nu_b$ factors through $\mathbf{D} \to \mathbb{G}_{\mathrm{m}}$ and the action of $s\nu_b(p)$ on $N_{\mathbf{p}}$ is a scalar. Thus, the slope homomorphism $\nu_b : \mathbf{D} \to G_p$ must be central. ∎

Remark 3.2. Lemma 3.1 provides a simple criterion for checking a polarized B-abelian variety $\underline{A} = (A, \lambda, \iota)$ being basic. Note that the assertion of the statement (2) depends only on the underlying structure of B-action, and not on the equipped polarization structure. Therefore, it makes sense to call a B-abelian variety (A, ι) *basic* if for any B-linear polarization λ on (A, ι), the polarized B-abelian variety (A, λ, ι) is basic in the sense of Definition 2.1. Such a polarization λ always exists; see Kottwitz [2, Lemma 9.2].

It follows from Lemma 3.1 that a B-abelian variety (A, ι) is basic if and only if the F-abelian variety $(A, \iota|_F)$ is basic, where $\iota|_F$ is the restriction of ι to F.

The following two lemmas are reorganized from [6, 6.26-6.29]; proofs are provided solely for the reader's convenience.

Lemma 3.3. *Given any set $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ of rational numbers with $0 \le \lambda_{\mathbf{p}} \le 1$ and $\lambda_{\mathbf{p}} + \lambda_{\bar{\mathbf{p}}} = 1$, there is a positive integer s and $u \in O_F[1/p]^\times$ such that*

$$u\bar{u} = q := p^s, \quad \textit{and} \quad \mathrm{ord}_{\mathbf{p}}\, u = s\lambda_{\mathbf{p}},\ \forall\, \mathbf{p} \in \Sigma_p. \tag{3.2}$$

Proof. Consider the map

$$\mathrm{ord} : O_F\left[\frac{1}{p}\right]^\times \to \bigoplus_{\mathbf{p}\in\Sigma_p} (1/e_{\mathbf{p}})\mathbb{Z}, \quad u \mapsto (\mathrm{ord}_{\mathbf{p}}(u))_{\mathbf{p}\in\Sigma_p},$$

where $e_{\mathbf{p}}$ is the ramification index of $\mathbf{p}$. By Dirichlet's unit theorem, the image has rank $|\Sigma_p|$ and is of finite index. Therefore, there are a positive integer s and an element $u \in O_F[1/p]^\times$ such that $\mathrm{ord}_{\mathbf{p}}(u) = s\lambda_{\mathbf{p}} =: r_{\mathbf{p}}$ for all $\mathbf{p} \in \Sigma_p$. Let $q = p^s$ and $u' := qu/\bar{u}$, then one computes

$$\mathrm{ord}_{\mathbf{p}}\, u' = 2r_{\mathbf{p}} \text{ and } u'\overline{u}' = q^2.$$

Replacing u by u' and q by q^2, one gets the desired result. ∎

The element u in Lemma 3.3 actually lies in O_F as $\mathrm{ord}_{\mathbf{p}}(u) \geq 0$ for all $\mathbf{p}|p$.

Lemma 3.4. *Fix $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ and $q = p^s$ as in Lemma 3.3, and an positive integer g. Then there is a positive integer n such that for any basic g-dimensional polarized O_B-abelian variety $\underline{A}$ over a finite extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ with slopes $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$, the n-th power of Frobenius morphism π_A^n lies in $\iota(F)$.*

Proof. We first prove that the statement holds for one such object $\underline{A}$, i.e. there is an integer n_A possibly depending on A such that $\pi_A^{n_A} \in \iota(F)$. Clearly the statement depends only on the isogeny class of $\underline{A}$. Let M be the Dieudonné module of $\underline{A}$. Within the isogeny class, we can choose $\underline{A}$ so that $\iota(O_F) \subset \mathrm{End}(A)$ and $F^s M_{\mathbf{p}} = p^{r_{\mathbf{p}}} M_{\mathbf{p}}$ for all $\mathbf{p} \in \Sigma_p$, where $r_{\mathbf{p}} = s\lambda_{\mathbf{p}}$ and $M = \oplus_{\mathbf{p}|p} M_{\mathbf{p}}$ is the decomposition with respect to (3.1). Let u be as in Lemma 3.3, then $\iota(u)^{-m}\pi_A$ is an automorphism of A that preserves the polarization as $\iota(u)^{-m}\pi_A(M_{\mathbf{p}}) = M_{\mathbf{p}}$ for all $\mathbf{p} \in \Sigma_p$. Therefore, a power of this automorphism is the identity by a theorem of Serre. Thus, a power of π_A is contained in $\iota(F)$.

Let $C := \mathrm{End}_B^0(A)$. As $\dim C$ is bounded by $4g^2$, there is a fixed positive integer n such that $\zeta^n = 1$ for any element $\zeta \in C$ of finite order. By the result we just proved that $\iota(u)^{-m}\pi_A \in C$ is of finite order, we have $\pi_A^n \in \iota(F)$ for all such objects $\underline{A}$. ∎

3.2. **Proof of Theorem 1.1.** Let the notation be as in Theorem 1.1. It suffices to show that A has smCM, that is, any maximal commutative semi-simple $\mathbb{Q}$-subalgebra of $\mathrm{End}^0(A)$ has degree $2\dim A$. Then by a theorem of Grothendieck (see a proof in [4] or [11]) there exists an abelian variety A' over a finite field κ and an isogeny $\varphi : A' \otimes_\kappa k \to A$ over k. Replacing A' by one in its isogeny class if necessary, we may assume that A' admits an action ι' of O_B so that the isogeny φ is O_B-linear. Take the pull-back polarization λ' on A', which is clearly defined over a finite field extension of κ.

Let $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ be the set of slopes for $\underline{A}$. Take $q = p^s$ and a positive integer n as in Lemmas 3.3 and 3.4. We can choose a field k_0 finitely generated over $\mathbb{F}_q$ over which $\underline{A}$ is defined. The abelian variety $\underline{A}$ extends to a polarized O_B-abelian scheme $\underline{\mathbf{A}}$ over $S = \mathrm{Spec}\, R$ for a finitely generated $\mathbb{F}_q$-subalgebra R of k_0 with fraction field $\mathrm{Frac}(R) = k_0$. We may assume further that S is smooth over $\mathrm{Spec}\, \mathbb{F}_q$. Let s be a closed point of S and η the generic point. By Grothendieck's specialization theorem, the special fiber $\underline{\mathbf{A}}_s$ over s also has the same slopes $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$, and hence is basic.

We identify the endomorphism rings $\mathrm{End}_{k_0}(A) = \mathrm{End}_R(\mathbf{A}) \subset \mathrm{End}(\mathbf{A}_{\bar{s}})$, and write ι for the O_B-actions on these abelian varieties. Let

$$\rho_\ell : \pi_1(S, \bar{\eta}) \to \mathrm{Aut}(T_\ell(A_{\bar{\eta}}))$$

be the associated ℓ-adic representation. The action of $\mathrm{Gal}(\bar{\eta}/\eta)$ on $T_\ell(A_{\bar{\eta}})$ factors through ρ_ℓ. Again we identify the Tate modules $T_\ell(\mathbf{A}_{\bar{s}}) = T_\ell(\mathbf{A}_{\widetilde{S}_{\bar{s}}}) = T_\ell(A_{\bar{\eta}})$, where $\widetilde{S}_{\bar{s}}$ is the (strict) Henselization of S at $\bar{s}$. Put $V_\ell(A_{\bar{\eta}}) := T_\ell(A_{\bar{\eta}}) \otimes \mathbb{Q}_\ell$.

Let π_{A_s} be the Frobenius morphism on $\mathbf{A}_s$ and Frob_s the geometric Frobenius element in $\pi_1(S,\bar\eta)$ corresponding to the closed point s. We have

(i) $\pi_{A_s}^n \in \iota(F) \subset \mathrm{End}(T_\ell(\mathbf{A}_{\bar s}))$, by Lemma 3.4;
(ii) $\rho_\ell(\mathrm{Frob}_s^n) = \pi_{A_s}^n$ lies in the center $Z(\mathbb{Q}_\ell)$ of $\mathrm{GU}_{B_\ell}(V_\ell(A_{\bar\eta}), \langle\, ,\rangle)$, by the identification of the Tate modules and (i);
(iii) the Frobenius elements Frob_s for all closed points s generate a dense subgroup of $\pi_1(S,\bar\eta)$.

Let $G_\ell := \rho_\ell(\pi_1(S,\bar\eta))$ be the ℓ-adic monodromy group. Let $m_n : G_\ell \to G_\ell$ be the map $x \mapsto x^n$. It is an open mapping and its image contains an open subgroup U of G_ℓ, which is of finite index. Clearly U lies in the center $Z(\mathbb{Q}_\ell)$ by (ii) and (iii). Replacing k_0 by a finite extension, we have $G_\ell \subset Z(\mathbb{Q}_\ell)$. Let $\mathbb{Q}_\ell[\pi]$ be the (commutative) subalgebra of $\mathrm{End}(V_\ell(A_{\bar\eta}))$ generated by G_ℓ. By Zarhin's theorem [16], $\mathbb{Q}_\ell[\pi]$ is semi-simple and commutative, and $\mathrm{End}_{\mathbb{Q}_\ell[\pi]}(V_\ell(A_{\bar\eta})) = \mathrm{End}(A)\otimes\mathbb{Q}_\ell$. Hence any maximal commutative semi-simple $\mathbb{Q}_\ell$-subalgebra of $\mathrm{End}(A)\otimes\mathbb{Q}_\ell$ is also a maximal one in $\mathrm{End}(V_\ell(A_{\bar\eta}))$. This shows that any maximal commutative semi-simple subalgebra of $\mathrm{End}^0(A)$ has degree $2g$, and hence completes the proof. ∎

3.3. **Consequences.** In [15] we defined a class of polarized B-abelian varieties, called of arithmetic type. For these abelian varieties the "simple mass formula" in [15, Theorem 2.2] remain valid for algebraically closed ground fields, not just for finitely generated fields over a prime field. We related these B-abelian varieties with basic B-abelian varieties in the case where the ground field k is $\overline{\mathbb{F}}_p$; see [15, Theorem 4.5]. Using Theorem 1.1, we extend this result to an arbitrary algebraically closed field k of characteristic $p>0$,

Recall that a polarized B-abelian variety (A,λ,ι) over an algebraically closed field k of characteristic $p>0$ is said to be *of arithmetic type* if there is a model (A_0,λ_0,ι_0) of (A,λ,ι) over a subfield k_0 finitely generated over $\mathbb{F}_p$ such that the associated Galois representation $\rho_\ell : \mathcal{G}_{k_0} \to \mathrm{GU}_B(V_\ell(A_0), e_{\lambda,\ell})$ (Section 2.1) is central for some prime $\ell \neq p$ (or equivalently, for all primes $\ell\neq p$, see [15, Proposition 3.10]). It is shown in [15, Section 3] that this is again a geometric notion which depends only on the underlying B-abelian variety (A,ι) and not on the carried polarization structure λ.

Theorem 3.5. *A B-abelian variety (A,ι) over an algebraically closed field k of characteristic $p>0$ is of arithmetic type if and only if it is basic.*

Proof. By Theorem 1.1, there is a B-abelian variety (A_0,ι_0) over $\overline{\mathbb{F}}_p$ and a B-linear isogeny $\varphi : (A_0,\iota_0)\otimes_{\overline{\mathbb{F}}_p} k \to (A,\iota)$. As a result we can reduce the statement to the case where $k=\overline{\mathbb{F}}_p$ and this is Theorem 4.5 of [15]. ∎

Proposition 3.6 (cf. [6, Corollary 6.29])**.** *Let K be a finite-dimensional semi-simple $\mathbb{Q}$-algebra that admits a positive involution. Let (A,ι) and (A',ι') be two basic K-abelian varieties over an algebraically closed field k of characteristic $p>0$. Then we have*

$$\mathrm{Hom}_K(A,A')\otimes_{\mathbb{Z}}\mathbb{Q}_\ell \simeq \mathrm{Hom}_K(V_\ell(A),V_\ell(A')) \quad \forall\, \ell\neq p, \tag{3.3}$$

and

$$\mathrm{Hom}_K(A,A')\otimes_{\mathbb{Z}}\mathbb{Q}_p \simeq \mathrm{Hom}_K((N,\mathcal{F}),(N',\mathcal{F})), \tag{3.4}$$

where N and N' are the isocrystals associated to (A,ι) and (A',ι'), respectively.

PROOF. Let E be the center of K. If (3.3) and (3.4) hold true where K is replaced by E, then (3.3) and (3.4) hold true. Note that A is a basic K-abelian variety if and only if it is a basic E-abelian variety (Remark 3.2) . Replacing K by its center, we may assume that K is commutative.

By Theorem 1.1, there are K-abelian varieties (A_0,ι_0) and (A'_0,ι'_0) over $\overline{\mathbb{F}}_p$ such that $(A_0,\iota_0)\otimes_{\overline{\mathbb{F}}_p} k\sim(A,\iota)$ and $(A'_0,\iota'_0)\otimes_{\overline{\mathbb{F}}_p} k\sim(A',\iota')$. We have a natural isomorphism

$$\mathrm{Hom}_K(A_0,A'_0)\otimes_{\mathbb{Z}}\mathbb{Q}\simeq\mathrm{Hom}_K(A,A')\otimes_{\mathbb{Z}}\mathbb{Q},$$

and natural identifications $V_\ell(A_0)=V_\ell(A)$ and $V_\ell(A'_0)=V_\ell(A')$ for $\ell\neq p$. For $\ell=p$, we have also the identification $\mathrm{Hom}_K((N_0,\mathcal{F}),(N'_0,\mathcal{F}))=\mathrm{Hom}_K((N,\mathcal{F}),(N',\mathcal{F}))$, where N_0 and N'_0 are the isocrystals associated to (A_0,ι_0) and (A'_0,ι'_0), respectively. Therefore, we are reduced to prove the statement where $k=\overline{\mathbb{F}}_p$, which is done by Rapoport-Zink (see [6, Corollary 6.29, p. 293]). ∎

Corollary 3.7. *Let (A,ι) and (A',ι') be two basic B-abelian varieties over an algebraically closed field k of characteristic p, with slopes $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ and $\{\lambda'_{\mathbf{p}}\}_{\mathbf{p}|p}$, respectively. Then (A,ι) and (A',ι') are isogenous if and only if $\lambda_{\mathbf{p}}=\lambda'_{\mathbf{p}}$ and* $\mathrm{rank}\,N_{\mathbf{p}}=\mathrm{rank}\,N'_{\mathbf{p}}$ *for all $\mathbf{p}|p$.*

PROOF. This follows from Proposition 3.6. ∎

4. A correspondence

4.1. Let $\mathbb{F}_q$ be the finite field of $q=p^s$ elements. Let $\mathcal{A}_{\mathbb{F}_q}$ denote the category of abelian varieties over $\mathbb{F}_q$ up to isogeny. Let $\mathcal{B}$ be the category defined as follows, which we call the category of basic abelian varieties with endomorphisms over $\overline{\mathbb{F}}_p$ up to isogeny. The objects of $\mathcal{B}$ consist of all triples (F,A,ι), where

- F is a finite-dimensional commutative semi-simple $\mathbb{Q}$-algebra that admits a positive involution, and
- (A,ι) is a basic F-abelian variety over $\overline{\mathbb{F}}_p$.

For any two objects $\underline{A}_1=(F_1,A_1,\iota_1)$ and $\underline{A}_2=(F_2,A_2,\iota_2)$ in $\mathcal{B}$, a morphism in $\mathrm{Hom}_{\mathcal{B}}(\underline{A}_1,\underline{A}_2)$ is a pair $(\varphi,\widetilde{\varphi})$, where

- $\widetilde{\varphi}:F_1\to F_2$ is a $\mathbb{Q}$-linear algebra homomorphism in a broader sense that the image $\widetilde{\varphi}(1_{F_1})$ of the identity 1_{F_1} may not be the identity 1_{F_2}, and
- φ is an element in $\mathrm{Hom}(A_1,A_2)\otimes\mathbb{Q}$ which is (F_1,F_2)-equivariant in the sense that $\varphi\circ\iota_1(a)=\iota_2(\widetilde{\varphi}(a))\circ\varphi$ for all $a\in F_1$.

Note that if the map $\widetilde{\varphi}:F_1\to F_2$ as above is surjective, then $\widetilde{\varphi}(1_{F_1})=1_{F_2}$ (as $\widetilde{\varphi}(1_{F_1})y=y$ for all $y\in F_2$), i.e. it is also a ring homomorphism. A reason we need to allow more general maps $\widetilde{\varphi}$ is as follows. Let (A_i,ι_i) be an F_i-abelian variety for $i=1,2$, and $\iota_1\times\iota_2:F_1\times F_2\to\mathrm{End}(A_1\times A_2)$ the product map. Then the map $\varphi=\mathrm{id}_{A_1}\times 0:A_1\to A_1\times A_2$ is an $(F_1,F_1\times F_2)$-equivariant with respect to the map $\widetilde{\varphi}=\mathrm{id}_{F_1}\times 0:F_1\to F_1\times F_2$. The latter map is not a ring homomorphism.

Clearly two objects $\underline{A}_1$ and $\underline{A}_2$ in $\mathcal{B}$ are isomorphic if and only if there is a $\mathbb{Q}$-algebra isomorphism $\widetilde{\varphi}:F_1\simeq F_2$, and an (F_1,F_2)-equivariant quasi-isogeny $\varphi:A_1\to A_2$ over $\overline{\mathbb{F}}_p$.

The category $\mathcal{B}$ is not yet good enough in comparison with the category of abelian varieties with fixed endomorphism structures; there are simply too many morphisms $\widetilde{\varphi}$ among the fields F. For example, when $F_1 = F_2 = F$, the usual notion of morphisms between two F-abelian varieties would require $\widetilde{\varphi}$ to be the identity and not an arbitrary automorphism as in the category $\mathcal{B}$.

We introduce another category $\mathcal{B}^{\rm rig}$, which we call the category of basic abelian varieties with rigidified endomorphisms over $\overline{\mathbb{F}}_p$ up to isogeny. The objects of $\mathcal{B}^{\rm rig}$ consist of all tuples (F, x, A, ι) over $\overline{\mathbb{F}}_p$, where (F, A, ι) is an object in $\mathcal{B}$ and $x \in F$ is an element generating F over $\mathbb{Q}$. Suppose (F, x, A, ι) is an object in $\mathcal{B}^{rig}$, let $\mathbb{Q}[t] \to F$ be the natural surjective map sending t to x, and $f : \mathbb{Q}[t] \to \mathrm{End}^0(A)$ be the morphism obtained by composing with the map ι. Given two objects $\underline{A}_i = (F_i, x_i, A_i, \iota_i)$ in $\mathcal{B}^{\rm rig}$ $(i = 1, 2)$, a morphism $\varphi : \underline{A}_1 \to \underline{A}_2$ in $\mathcal{B}^{\rm rig}$ is an element $\varphi \in \mathrm{Hom}(A_1, A_2) \otimes \mathbb{Q}$ such that $\varphi \circ f_1(a) = f_2(a) \circ \varphi$ for all $a \in \mathbb{Q}[t]$, where $f_i : \mathbb{Q}[t] \to \mathrm{End}^0(A_i)$ are the maps associated as above. In the case when $F_1 = F_2 = F$, we have

$$\mathrm{Hom}_F((A_1, \iota_1), (A_2, \iota_2)) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}_{\mathcal{B}^{\rm rig}}((F, x, A_1, \iota_1), (F, x, A_2, \iota_2))$$

for any element x generates F over $\mathbb{Q}$, which recovers the usual notion of morphisms of F-abelian varieties (though we may not really want the additional structure x).

We shall embed $\mathcal{A}_{\mathbb{F}_q}$ as a full subcategory of $\mathcal{B}^{\rm rig}$. As the first step, we prove the following result.

Proposition 4.1. *Let A be an abelian variety over $\mathbb{F}_q$ and π_A its Frobenius endomorphism. Put $F := \mathbb{Q}(\pi_A)$ and $\iota : F \to \mathrm{End}^0(A)$ for the inclusion. Then the F-abelian variety (A, ι) is basic.*

PROOF. Suppose that the finite field k has $q = p^s$ elements. Let $A \sim \Pi_{i=1}^t A_i^{n_i}$ be the decomposition into components up to isogeny, where each abelian variety A_i is simple and $A_i \not\sim A_j$ for any $i \neq j$. Let π_i be the Frobenius endomorphism of A_i and put $F_i := \mathbb{Q}(\pi_i)$. Then we have $F = \prod_i^t F_i$. Let $\Sigma_{p,i}$ be the set of the primes $\mathbf{p}$ of F_i over p. Thus, Σ_p is the disjoint union of $\Sigma_{p,i}$ for $i = 1, \dots, t$. Let N (resp. N_i) be the isocrystal associated to the F-abelian variety $\underline{A} = (A, \iota)$ (resp. $\underline{A}_i = (A_i, \iota_i)$). Clearly if $\mathbf{p} \in \Sigma_{p,i}$ then $N_{\mathbf{p}} = N_{i,\mathbf{p}}^{n_i}$. In particular, $N_{\mathbf{p}}$ is isotypic for all $\mathbf{p} \in \Sigma_p$ if and only if $N_{i,\mathbf{p}}$ is isotypic for all i and all $\mathbf{p} \in \Sigma_{p,i}$. It follows from Lemma 3.1 that $\underline{A}$ is basic if and only if $\underline{A}_i$ is basic for all $i = 1, \dots, t$. Therefore, it suffices to prove the statement when A is simple. In this case, as $F^s = \pi$ and $\pi \in F_{\mathbf{p}}$, the component $N_{\mathbf{p}}$ has slope $\mathrm{ord}_{\mathbf{p}}(\pi)/s$. ∎

By Lemma 3.1, if K is any commutative semi-simple $\mathbb{Q}$-subalgebra of the endomorphism algebra $\mathrm{End}^0(A)$ which is stable under a Rosati involution and contains F, then (A, i) with $i : K \subset \mathrm{End}^0(A)$, is also a basic K-abelian variety. Our way of making A into a basic abelian variety with endomorphism structures as in Proposition 4.1 is, after a suitable base change, the most "economical" one. Namely, one uses the least endomorphisms.

Proposition 4.2. *Let A be an abelian variety over $\mathbb{F}_q$ such that* $\mathrm{End}(A) = \mathrm{End}(\overline{A})$, *where $\overline{A} = A \otimes \overline{\mathbb{F}}_p$. Suppose that K is a commutative semi-simple $\mathbb{Q}$-algebra admitting a positive involution, and (A, ι) is a basic K-abelian variety. Then $\iota(K)$ contains the center F of the endomorphism algebra* $\mathrm{End}^0(A)$.

PROOF. Let π be the Frobenius endomorphism of A. Then for any positive integer n one has $F=\mathbb{Q}(\pi^n)$ as F is the center of the endomorphism algebra $\mathrm{End}^0(A\otimes_{\mathbb{F}_q}\mathbb{F}_{q^n})$. Now using Lemma 3.4, there is a positive integer n such that π^n is contained in $\iota(K)$. As a result, the center F is contained in $\iota(K)$. ∎

Now we define a functor $\Phi:\mathcal{A}_{\mathbb{F}_q}\to\mathcal{B}^{\mathrm{rig}}$ as follows. To each abelian variety A over $\mathbb{F}_q$ we associate a tuple $(F,\pi_A,\overline{A},\iota)$, where π_A is the Frobenius endomorphism of A, $F:=\mathbb{Q}(\pi_A)$, $\overline{A}:=A\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_p$ and $\iota:F\to\mathrm{End}^0(\overline{A})$ is the inclusion. Clearly we have the associated map

$$\Phi_*:\mathrm{Hom}(A_1,A_2)\otimes\mathbb{Q}\to\mathrm{Hom}_{\mathcal{B}^{\mathrm{rig}}}(\Phi(A_1),\Phi(A_2)) \tag{4.1}$$

as $\varphi\circ\iota_1(\pi_{A_1})=\iota_2(\pi_{A_2})\circ\varphi$ for any map $\varphi\in\mathrm{Hom}(A_1,A_2)\otimes\mathbb{Q}$.

Theorem 4.3. *The functor $\Phi:\mathcal{A}_{\mathbb{F}_q}\to\mathcal{B}^{\mathrm{rig}}$ is fully faithful.*

PROOF. Let A_1 and A_2 be two abelian varieties over $\mathbb{F}_q$, and let $\underline{A}_i:=(F_i,\pi_i,\overline{A}_i,\iota_i)$ be the associated object in $\mathcal{B}^{\mathrm{rig}}$ for $i=1,2$. We must show that the associated map Φ_* in (4.1) is bijective. It is clear that Φ_* is injective. Let $\overline{f}:\overline{A}_1\to\overline{A}_2$ be an element in $\mathrm{Hom}_{\mathcal{B}^{\mathrm{rig}}}(\Phi(A_1),\Phi(A_2))$, particularly $\pi_2\overline{f}=\overline{f}\pi_1$. As $\sigma_q(\overline{f})=\pi_2\overline{f}\pi_1^{-1}=\overline{f}$, where $\sigma_q\in\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$ is the Frobenius map, the morphism $\overline{f}$ is defined over $\mathbb{F}_q$. ∎

4.2. We restrict the functor Φ to the objects for which the endomorphism algebras have a common center. Fix any abelian variety A_0 over $\mathbb{F}_q$. Let π_0 be the Frobenius endomorphism of A_0 over $\mathbb{F}_q$, $p(t)\in\mathbb{Z}[t]$ its minimal polynomial over $\mathbb{Q}$ and $F:=\mathbb{Q}[t]/(p(t))$. A commutative semi-simple $\mathbb{Q}$-algebra F arising in this way is called a *q-Weil $\mathbb{Q}$-algebra.*

Let $\mathcal{A}_{\pi_0,\mathbb{F}_q}$ denote the full subcategory of $\mathcal{A}_{\mathbb{F}_q}$ consisting of all abelian varieties A such that the minimal polynomial of the Frobenius endomorphism of A is equal to $p(t)$. In other words, every abelian variety A over $\mathbb{F}_q$ in $\mathcal{A}_{\pi_0,\mathbb{F}_q}$ shares the same simple components of A_0 up to isogeny.

Let $\mathcal{B}_F$ denote the category of basic F-abelian varieties over $\overline{\mathbb{F}}_p$ up to isogeny. Similarly we define a functor

$$\Phi_F:\mathcal{A}_{\pi_0,\mathbb{F}_q}\to\mathcal{B}_F,\quad A\mapsto(\overline{A},\iota), \tag{4.2}$$

where $\overline{A}:=A\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_p$ and $\iota:F\to\mathrm{End}^0(\overline{A})$ is the ring monomorphism sending t to π_A. By Theorem 4.3, we obtain the following result.

Proposition 4.4. *For any q-Weil $\mathbb{Q}$-algebra $F=\mathbb{Q}(\pi_0)$, the functor $\Phi_F:\mathcal{A}_{\pi_0,\mathbb{F}_q}\to\mathcal{B}_F$ is fully faithful.*

Remark 4.5. The functor Φ_F is usually not essentially surjective. For example take $q=p^2$ and $\pi_0=p\zeta_6$ with $p\equiv 1 \pmod 3$. The corresponding abelian variety A_0 is a simple supersingular abelian surface, and any object in $\mathcal{A}_{\pi_0,\mathbb{F}_q}$ is isogenous to a finite product of copies of A_0. However, as $F=\mathbb{Q}(\sqrt{-3})$ and p splits in F, there is an ordinary elliptic curve E over $\overline{\mathbb{F}}_p$ and there is an isomorphism $i:F\simeq\mathrm{End}^0(E)$. The F-elliptic curve (E,i) is clearly in $\mathcal{B}_F$ but is not isogenous to a finite product of copies of A_0. In this case the functor Φ_F is not essentially surjective. A point is that different Weil numbers can generate the same field.

5. A mass formula

5.1. **Within a simple isogeny class.** Let π be a Weil q-number, $F=\mathbb{Q}(\pi)$ the number field generated by π over $\mathbb{Q}$, and O_F the ring of integers in F. Let $\mathrm{Isog}(\pi)$ denote the simple isogeny class corresponding to π by the Honda-Tate theory [8]. Let A_0 be an abelian variety over $\mathbb{F}_q$ in $\mathrm{Isog}(\pi)$ and put $d:=\dim(A_0)$.

Let $\Lambda(\pi)$ denote the set of isomorphism classes of abelian varieties over $\mathbb{F}_q$ in $\mathrm{Isog}(\pi)$, and $\Lambda(\pi)^{\max}\subset\Lambda(\pi)$ be the subset consisting of all abelian varieties A such that the ring O_F is contained in $\mathrm{End}(A)$. Let $\mathbf{B}_{d,O_F}$ denote the set of isomorphism classes of d-dimensional basic O_F-abelian varieties over $\overline{\mathbb{F}}_p$.

The following lemma follows from Proposition 4.4.

Lemma 5.1. *The association $A\mapsto(\overline{A},\iota)$ induces an injective map $\Phi_\pi:\Lambda(\pi)^{\max}\to\mathbf{B}_{d,O_F}$.*

If $A\in\Lambda(\pi)^{\max}$ is an abelian variety over $\mathbb{F}_q$ and $(\overline{A},\iota)$ the corresponding basic O_F-abelian variety over $\overline{\mathbb{F}}_p$, then clearly any O_F-linear polarization $\overline{\lambda}$ on $(\overline{A},\iota)$ descends uniquely to a polarization λ on A over $\mathbb{F}_q$. Particularly, the map $\lambda\mapsto\overline{\lambda}$ gives rise to a one-to-one correspondence between polarizations on A and O_F-linear polarizations on $(\overline{A},\iota)$ over $\overline{\mathbb{F}}_p$. It follows that A admits a principal polarization if and only if $(\overline{A},\iota)$ admits a principal O_F-linear polarization. Moreover, we also have a natural isomorphism of finite groups

$$\mathrm{Aut}(A,\lambda)\simeq\mathrm{Aut}(\overline{A},\overline{\lambda},\iota). \tag{5.1}$$

Now we let $\Lambda(\pi)_1^{\max}$ be the set of isomorphism classes of principally polarized abelian varieties (A,λ) over $\mathbb{F}_q$ such that the underlying abelian variety A belongs to $\Lambda(\pi)^{\max}$. The set $\Lambda(\pi)_1^{\max}$ could be empty; nevertheless, it is always finite. This follows from the finiteness of the set $\mathcal{A}_{d,1}(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of the Siegel modular variety $\mathcal{A}_{d,1}$,

Let $\mathcal{A}_{d,O_F,1}$ be the moduli space over $\overline{\mathbb{F}}_p$ of d-dimensional principally polarized O_F-abelian varieties, and $\mathbf{B}_{d,O_F,1}\subset\mathcal{A}_{d,O_F,1}(\overline{\mathbb{F}}_p)$ be its basic locus. Then the map Φ_π induces an injective map

$$\Phi_\pi:\Lambda(\pi)_1^{\max}\to\mathbf{B}_{d,O_F,1}. \tag{5.2}$$

We have the following commutative diagram

$$\begin{array}{ccc}\Lambda(\pi)_1^{\max} & \xrightarrow{\Phi_\pi} & \mathbf{B}_{d,O_F,1}\\ \downarrow & & \downarrow\\ \Lambda(\pi)^{\max} & \xrightarrow{\Phi_\pi} & \mathbf{B}_{d,O_F},\end{array}$$

where the vertical maps forget the polarization.

The mass of $\Lambda(\pi)_1^{\max}$ is defined as

$$\mathrm{Mass}(\Lambda(\pi)_1^{\max}):=\sum_{(A,\lambda)\in\Lambda(\pi)_1^{\max}}|\mathrm{Aut}(A,\lambda)|^{-1} \tag{5.3}$$

if it is nonempty, and to be zero otherwise. Similarly, any finite subset $S\subset\mathcal{A}_{d,O_F,1}(\overline{\mathbb{F}}_p)$, the mass of S is defined as

$$\mathrm{Mass}(S):=\sum_{(\overline{A},\overline{\lambda},\iota)\in S}|\mathrm{Aut}(\overline{A},\overline{\lambda},\iota)|^{-1} \tag{5.4}$$

if S is nonempty and $\mathrm{Mass}(S)=0$ otherwise. It follows from (5.1) that

$$\mathrm{Mass}(\Lambda(\pi)_1^{\max}) = \mathrm{Mass}(\mathrm{Im}\Phi_\pi). \tag{5.5}$$

5.2. **An example with $\pi=\sqrt{p}$.** We consider a special case of the previous construction when $\pi=\sqrt{p}$. The result we obtain is the following.

Theorem 5.2. *Let $\pi=\sqrt{p}$. Then the finite set $\Lambda(\pi)_1^{\max}$ is nonempty and we have*

$$\mathrm{Mass}(\Lambda(\pi)_1^{\max} = \frac{1}{4}\,\zeta_{\mathbb{Q}(\sqrt{p})}(-1). \tag{5.6}$$

We need a general result.

Proposition 5.3. *Let F be a totally real field, $\mathcal{O}:=O_F\otimes_{\mathbb{Z}}\mathbb{Z}_p$ and k an algebraically closed field of characteristic $p>0$.*

(1) *Let $\underline{M}=(M,\langle\,,\rangle,\iota_M)$ be a supersingular separably quasi-polarized Dieudonné $\mathcal{O}$-module over k satisfying the following condition*

$$(*)\quad \mathrm{tr}(\iota_M(a))\cdot[F:\mathbb{Q}] = (\mathrm{rank}_W M)\cdot \mathrm{tr}_{F/\mathbb{Q}}(a), \quad \forall\, a\in O_F.$$

Then there is a supersingular principally polarized O_F-abelian variety $\underline{A}=(A,\lambda,\iota)$ over k whose Dieudonné module $M(\underline{A})$ is isomorphic to $\underline{M}$.

(2) *Assume that p is totally ramified in F. Then for any supersingular Dieudonné $\mathcal{O}$-module $\underline{M}=(M,\iota_M)$ over k of W-rank $2[F:\mathbb{Q}]$, there is a principally polarized O_F-abelian variety $\underline{A}=(A,\lambda,\iota)$ over k such that the Dieudonné $\mathcal{O}$-module $M(A,\iota)$ is isomorphic to $\underline{M}$.*

Proof. (1) By [13, Theorem 1.1], there is a (prime-to-p degree) polarized O_F-abelian variety $\underline{A}=(A,\lambda,\iota)$ such that $M(\underline{A})\simeq\underline{M}$. We can choose a self-dual $(O_F\otimes\mathbb{Z}_\ell)$-lattice L_ℓ in $V_\ell(A)$ with respect to $e_{\lambda,\ell}$ for each prime $\ell\neq p$ with $L_\ell=T_\ell(A)$ for almost all ℓ. The proof of the existence of such a lattice L_ℓ is elementary and left to the reader. Then there is an O_F-abelian variety (A',ι') and a prime-to-p degree O_F-linear quasi-isogeny $\varphi':(A',\iota')\to(A,\iota)$ such that $\varphi'_*(T_\ell(A'))=L_\ell$ for all $\ell\neq p$. Then the pull-back $\lambda':=\varphi^*\lambda$ by φ is a principal polarization as L_ℓ is self-dual. The object (A',λ',ι') is a desired one.

(2) Since there is only one prime of O_F over p, the condition $(*)$ is satisfied. By [10, Proposition 2.8], the Dieudonné $\mathcal{O}$-module $\underline{M}$ admits a separable $\mathcal{O}$-linear quasi-polarization, noting that an equivalent condition (5) of loc. cit. is satisfied when p is totally ramified. Then the statement follows from (1). ∎

Now we return to our case $F=\mathbb{Q}(\sqrt{p})$, where $\mathcal{O}=O_F\otimes\mathbb{Z}_p=\mathbb{Z}_p[\sqrt{p}]$. The prime p is ramified in F with ramification index $e=2$. Clearly any member A in $\Lambda(\pi)^{\max}$ is a superspecial abelian surface over $\mathbb{F}_p$. The Dieudonné module $M=M(A)$ of A is a rank 4 free $\mathbb{Z}_p$-module together with a $\mathbb{Z}_p$-linear action by O_F. Therefore, $M\simeq\mathcal{O}^2$ on which both the Frobenius $\mathcal{F}$ and the Verschiebung $\mathcal{V}$ operate by $\sqrt{p}$. From this the Lie algebra $\mathrm{Lie}(A)=M/\mathcal{V}M$ of A is isomorphic to $\mathbb{F}_p\oplus\mathbb{F}_p$ as an (O_F/p)-module. In other words, A has Lie type $(1,1)$ in the terminology of [10, Section 1]. Therefore, the injective map $\Phi_\pi:\Lambda(\pi)^{\max}\to\mathbf{B}_{2,O_F}$ factors through the subset $\mathbf{S}\subset\mathbf{B}_{2,O_F}$ of superspecial abelian O_F-surfaces of Lie type $(1,1)$.

We first claim that the induced map

$$\Phi_\pi:\Lambda(\pi)^{\max}\to\mathbf{S} \tag{5.7}$$

is bijective. Fix a member $A_0 \in \Lambda(\pi)^{\max}$. By Waterhouse [9, Theorem 6.2], there is a natural bijection between the set $\Lambda(\pi)^{\max}$ and the set $\mathrm{Cl}(\mathrm{End}(A_0))$ of right ideal classes. Since the map Φ_π is injective, it suffices to show that $\mathbf{S}$ has the same cardinality as $\mathrm{Cl}(\mathrm{End}(A_0))$. Note that the isomorphism classes of (unpolarized) *superspecial* Dieudonné $\mathcal{O}$-modules are uniquely determined by their Lie types [12, Lemma 3.1]. It follows that the Dieudonné modules and Tate modules of any two members in $\mathbf{S}$ are mutually isomorphic (compatible with the actions of O_F). By (the unpolarized variant of) [12, Theorem 2.1], there is a natural bijection $\mathbf{S} \simeq \mathrm{Cl}(\mathrm{End}_{O_F}(\overline{A}_0))$. Since we have $\mathrm{End}(A_0) = \mathrm{End}_{O_F}(\overline{A}_0)$, our claim is proved.

Let $\mathbf{S}_1 \subset \mathbf{B}_{2,O_F,1}$ be the subset consisting of objects (A, λ, ι) so that the underlying abelian O_F-surface (A, ι) belongs to $\mathbf{S}$. Proposition 5.3 implies that $\mathbf{S}_1$ is nonempty. Consider the commutative diagram

$$\begin{array}{ccc} \Lambda(\pi)^{\max}_1 & \xrightarrow{\Phi_\pi} & \mathbf{S}_1 \\ {\scriptstyle f_\Lambda}\downarrow & & {\scriptstyle f_{\mathbf{S}}}\downarrow \\ \Lambda(\pi)^{\max} & \xrightarrow[\simeq]{\Phi_\pi} & \mathbf{S} \end{array} \tag{5.8}$$

Note that a member A in $\Lambda(\pi)^{\max}$ admits a principal polarization if and only if $\Phi_\pi(A) = (\overline{A}, \iota)$ admits a principal O_F-linear polarization. Moreover, the equivalence classes of principal polarizations on A are in bijection with the equivalence classes of principal O_F-linear polarizations on $(\overline{A}, \iota)$. It follows that the diagram (5.8) is cartesian, which particularly implies that the map $\Phi_\pi : \Lambda(\pi)^{\max}_1 \simeq \mathbf{S}_1$ is an isomorphism. Thus, we have proved $\mathrm{Mass}(\Lambda(\pi)^{\max}_1) = \mathrm{Mass}(\mathbf{S}_1)$.

Now we use the mass formula for $\mathrm{Mass}(\mathbf{S}_1)$ [12, Theorem 3.7]*

$$\mathrm{Mass}(\Lambda(\pi)^{\max}_1) = \mathrm{Mass}(\mathbf{S}_1) = \frac{1}{4}\zeta_F(-1); \tag{5.9}$$

this proves Theorem 5.2.

5.3. **Fibers of the map $f_{\mathbf{S}}$.** We describe the fibers of the map $f_{\mathbf{S}}$ in (5.8). Suppose (A, λ_0, ι) is a member in $\mathbf{S}_1$. Put $D := \mathrm{End}^0_{O_F}(A)$ and $O_D := \mathrm{End}_{O_F}(A)$. Then D is the quaternion F-algebra ramified only at the two real places of F and O_D is a maximal order. Note that the canonical involution $'$ is the unique positive involution on D. Therefore the Rosati involution induced by any O_F-linear polarization must be $'$. Suppose λ is another O_F-linear principal polarization, then $\lambda = \lambda_0 a$ for some totally positive symmetric element $a \in O_D^\times$, so $a \in O_{F,+}^\times$, the set of totally positive units in O_F. Suppose $b \in \mathrm{Aut}_{O_F}(A)$ is an O_F-linear automorphism. Then the pull-back

$$b^*(\lambda_0 a) = b^t \lambda_0 a b = \lambda_0 \lambda_0^{-1} b^t \lambda_0 b a = \lambda_0 (b' b) a.$$

Therefore, the set of equivalence classes of principal O_F-linear polarizations on (A, ι) is in bijection with the set $O_{F,+}^\times / \mathrm{Nr}(O_D^\times)$, where $\mathrm{Nr} : O_D \to O_F$ is the reduced norm. In other words, we obtain an isomorphism

$$f_{\mathbf{S}}^{-1}(A, \iota) \simeq O_{F,+}^\times / \mathrm{Nr}(O_D^\times). \tag{5.10}$$

*There is an error in the computation of the mass formula there. The error occurs in Lemma 3.4 of loc. cit., where the unramified quadratic order $O_{\mathbf{F}'_\mathfrak{p}}$ of $O_{\mathbf{F}_\mathfrak{p}}$ cannot be written as $O_{\mathbf{F}_\mathfrak{p}}[\sqrt{c}]$ when $p = 2$ as stated. As a result, the order A_ϵ when $\epsilon = 0$ as in Lemma 3.4 should be maximal, and the term $o_\mathfrak{p}$ should be always one in that paper, particularly in the formulas of Theorems 4.4 and 4.5.

As $\mathrm{Nr}(O_D^\times) \supset (O_F^\times)^2$, the group $O_{F,+}^\times / \mathrm{Nr}(O_D^\times)$ is a homomorphism image of $O_{F,+}^\times/(O_F^\times)^2$. The latter group has 1 or 2 elements according as the fundamental unit ϵ of F has norm -1 or not. Therefore, if $N(\epsilon) = -1$, then $f_{\mathbf{S}}^{-1}(A, \iota)$ has one element. Otherwise, the fiber $f_{\mathbf{S}}^{-1}(A, \iota)$ has at most two elements.

Acknowledgments

Theorem 1.1 is a result of an old (unpublished) preprint [14] which was prepared during the author's stay in the Max-Planck-Institut für Mathematik in Bonn. The author is grateful to Ching-Li Chai for the idea of the proof of Theorem 1.1 and thanks the MPIM for its kind hospitality and excellent conditions. The author thanks Jiangwei Xue for his patient clarifications and helpful comments on a revised version of the manuscript. The author is partially supported by the grants MoST 100-2628-M-001-006-MY4 and 104-2115-M-001-001-MY3. He also thanks the referee for careful reading and helpful comments which improve the exposition of the manuscript.

References

[1] R. E. Kottwitz, Isocrystals with additional structure. *Compositio Math.* **56** (1985), 201–220.
[2] R. E. Kottwitz, Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.* **5** (1992), 373–444.
[3] D. Mumford, *Abelian Varieties.* Oxford University Press, 1974.
[4] F. Oort, The isogeny class of a CM-type abelian variety is defined over a finite extension of the prime field. *J. Pure Appl. Algebra* **3** (1973), 399–408.
[5] M. Rapoport, On the Newton stratification. *Sém. Bourbaki Exp.* 903 (2001/2002), 207–224, *Astériques* No. 290, 2003.
[6] M. Rapoport and Th. Zink, *Period Spaces for p-divisible groups.* Ann. Math. Studies 141, Princeton Univ. Press, 1996.
[7] J. Tate, Endomorphisms of abelian varieties over finite fields. *Invent. Math.* **2** (1966), 134–144.
[8] J. Tate, Classes d'isogenie de variétés abéliennes sur un corps fini (d'après T. Honda). *Sém. Bourbaki Exp.* 352 (1968/69). Lecture Notes in Math., vol. 179, Springer-Verlag, 1971.
[9] W. C. Waterhouse, Abelian varieties over finite fields. *Ann. Sci. École Norm. Sup. (4)* **2** (1969), 521–560.
[10] C.-F. Yu, On reduction of Hilbert-Blumenthal varieties. *Ann. Inst. Fourier (Grenoble)* **53** (2003), 2105–2154.
[11] C.-F. Yu, The isomorphism classes of abelian varieties of CM-type. *J. Pure Appl. Algebra* **187** (2004) 305–319.
[12] C.-F. Yu, On the mass formula of supersingular abelian varieties with real multiplications. *J. Australian Math. Soc.* **78** (2005), 373–392.
[13] C.-F. Yu, On the slope stratification of certain Shimura varieties. *Math. Z.* **251** (2005), 859–873.
[14] C.-F. Yu, Basic points in the moduli spaces of PEL-type. MPIM-preprint 2005 – 103, 11 pp.
[15] C.-F. Yu, Simple mass formulas on Shimura varieties of PEL-type. *Forum Math.* **22** (2010), no. 3, 565–582.
[16] J.G. Zarhin, Isogenies of abelian varieties over fields of finite characteristics, *Math. USSR Sbornik* **24** (1974), 451–461.

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